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The geometry and significance of weak energy

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Abstract. The theory of weak values for quantum mechanical observables has come to serve as a useful basis for contemporary discussions concerning such varied topics as the tunnelling-time controversy and quantum stochastic processes. An intrinsic complex-valued weak energy has recently been observed experimentally and reported in the literature. In this paper it is shown that: (a) the real and imaginary valued parts of this weak energy have geometric interpretations related to a phase acquired from parallel transport in Hilbert space and the variational dynamics occurring in the associated projective Hilbert space, respectively; (b) the weak energy defines functions which translate correlation amplitudes and probabilities in time; (c) correlation probabilities can be controlled by manipulating the weak energy and there exists a condition of weak stationarity that guarantees their time invariance; and (d) a time-weak energy uncertainty relation of the usual form prevails when a suitable set of dynamical constraints are imposed.

1. Introduction

The use of pre-selection and post-selection measurement techniques for the control and manipulation of quantum physical systems was first introduced by Schrödinger [1, 2] over half a century ago. As a result of the recent advent of precision experimental instrumentation, there has been a renewed theoretical and utilitarian interest in such methods [3–6]. An especially interesting related application, the notion of the 'weak value' of a quantum mechanical observable, has been introduced by Aharonov *et al* [7, 8] and Aharonov and Vaidman [9] (hereafter referred to collectively as AAV). This value is the statistical result of a standard measurement procedure upon a pre-selected and post-selected ensemble of quantum systems when the interaction between the measuring apparatus and the system is sufficiently weak.

AAV's analyses led to the controversial result that such a measurement procedure can yield values for an observable that lie well outside its associated eigenvalue spectrum [10, 11]. This controversy was resolved in a theoretical sense by Duck *et al* [12] (hereafter referred to as DSS), who showed that these results were completely consistent with conventional interpretations of quantum mechanics. Since then, weak values have been discussed from a foundational perspective in the context of elements of reality [13] and non-locality [14]; have been associated with conditional probabilities in order to study the tunnelling-time controversy [15, 16]; and have been applied to the description of quantum stochastic processes [17].

Experimental measurements of weak values of the photon polarization state have been reported recently [18]. These results not only confirm the theory outlined by DSS, but also show theoretically that there is an intrinsic complex-valued weak energy that appears in the equation of motion for the weak value of an observable when the associated pre-selected and

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post-selected states are explicitly time dependent. This weak energy has been observed in the experimental data.

In this paper, this weak energy is discussed from geometric and dynamical perspectives. In particular, it is shown that: (a) the imaginary part of the weak energy provides for a variational description of the dynamics of a pre-selected/post-selected system in projective Hilbert space in terms of a generalized Fubini–Study metric; (b) the real part of the weak energy is the time rate of change of a phase acquired by parallel transporting the post-selected state to the ray containing the pre-selected state along the shortest geodesic joining their images in projective Hilbert space; (c) the weak value for a composite evolution operator associated with the pre-selected and post-selected states defines functions of the weak energy which translate in time the correlation amplitude and probability for the system; (d) correlation probabilities can be controlled by manipulating the weak energy and a weak stationarity condition exists for which correlation probabilities are time invariant; and (e) a time-weak energy uncertainty relation of the usual form prevails when a suitable set of dynamical constraints are imposed. A simple two state system and the 'dynamic quantum eraser' are used to illustrate these results.

2. A geometric interpretation

Following Parks *et al* [18], the complex-valued weak energy $(H' - H)_w$ is defined as

$$(H' - H)_w \equiv \frac{\langle \Psi'(t + \Delta t)|\hat{H}' - \hat{H}|\Psi(t)\rangle}{\langle \Psi'(t + \Delta t)|\Psi(t)\rangle} \tag{1}$$

where $|\Psi\rangle$ and $|\Psi'\rangle$ are the non-orthogonal time-dependent normalized pre-selected and post-selected states, respectively, with

$$\mathrm{i}\hbar \frac{\mathrm{d}|\Psi(t)\rangle}{\mathrm{d}t} = \hat{H}|\Psi(t)\rangle \qquad \text{and} \qquad \mathrm{i}\hbar \frac{\mathrm{d}\langle \Psi'(t')|}{\mathrm{d}t'} = -\langle \Psi'(t')|\hat{H}'. \tag{2}$$

Here $t' = t + \Delta t$ and Δt is the constant difference between the times of the pre-selection and post-selection measurements.

Let $\wp$ be the projective Hilbert space for this quantum system consisting of all the rays for the associated Hilbert space $\mathcal{H}$. Recall that a ray is an equivalence class $[\varphi]$ of states $|\varphi\rangle$ in $\mathcal{H}$ which differ only in phase. There is thus an induced projection map $\Pi : \mathcal{H} \to \wp$ such that $|\varphi\rangle \mapsto [\varphi]$. The evolution of the pre-selected/post-selected quantum system above is therefore represented geometrically by two curves in $\mathcal{H}$ such that at any time t, $|\Psi\rangle$ and $|\Psi'\rangle$ in $\mathcal{H}$ have images $[\Psi] = \Pi(|\Psi\rangle)$ and $[\Psi'] = \Pi(|\Psi'\rangle)$ in $\wp$. The distance separating $p = [\Psi]$ and $p' = [\Psi']$ in $\wp$ can be expressed in terms of their pre-images under Π using the generalized Fubini–Study metric [19–22]

$$s^2 = s(p', p)^2 = 4(1 - |\langle \Psi'|\Psi\rangle|^2) \tag{3}$$

where, in general, $|\Psi\rangle$ and $|\Psi'\rangle$ are any normalized states contained in the rays p and p', respectively.

When p and p' refer to rays containing the pre-selected and post-selected states, respectively, then straightforward rearrangement of the time derivative of (3) provides the important identity

$$L(s, \dot{s}) = \mathrm{Im}(H' - H)_w = \left\{ \frac{\hbar s}{(4 - s^2)} \right\} \left(\frac{\mathrm{d}s}{\mathrm{d}t} \right). \tag{4}$$

Here, if s is identified as the generalized coordinate, then $L(s, \dot{s})$ satisfies the associated *Euler–Lagrange* equation. Hence, s is such that the action

$$\Gamma = \int_{t_1}^{t_2} L(s, \dot{s})\,\mathrm{d}t$$

is an extremum as $|\Psi\rangle$ and $|\Psi'\rangle$ change with time. The imaginary part of the weak energy therefore provides a variational description of the dynamics of a pre-selected/post-selected system in $\wp$ in terms of the metric given by (3).

It is interesting to note from (4) that $L(s, \dot{s}) \to \infty$ is possible as $|\Psi\rangle$ and $|\Psi'\rangle$ become orthogonal. Also, if the distance s between $|\Psi\rangle$ and $|\Psi'\rangle$ remains constant with time, then $(\mathrm{d}s/\mathrm{d}t) = 0$ and $L(s, \dot{s}) = 0$. This condition clearly occurs when $|\Psi\rangle = |\Psi'\rangle$, since $s = 0$. Indeed, in this case $\hat{H}' = \hat{H}$ so that (1) vanishes. Thus, a non-vanishing weak energy can only exist when the time evolution of the pre-selected and post-selected states are governed by distinct system Hamiltonians.

Anandan and Aharonov [23] have shown that the Pancharatnam phase [24] χ defined by

$$\mathrm{e}^{\mathrm{i}\chi} \equiv \frac{\langle\Psi'|\Psi\rangle}{|\langle\Psi'|\Psi\rangle|} \qquad \langle\Psi'|\Psi\rangle \neq 0 \tag{5}$$

is the phase difference between $|\Psi\rangle$ and $|\varphi\rangle$, where $|\varphi\rangle$ is the state contained in $[\Psi]$ obtained by parallel transporting $|\Psi'\rangle$ along the unique path in $\mathcal{H}$ which is the pre-image under Π of the shortest geodesic joining $[\Psi']$ and $[\Psi]$ in $\wp$ (the Pancharatnam phase has also been discussed from a quantum mechanical perspective by Samuel and Bhandari [25]). If $|\Psi\rangle$ and $|\Psi'\rangle$ are time-dependent pre-selected and post-selected states, respectively, then rearrangement of the time derivative of (5), along with application of (2), readily yields

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = \left(\frac{1}{\hbar}\right) \mathrm{Re}(H' - H)_w. \tag{6}$$

Thus, the real part of the weak energy provides the rate of change for the phase acquired by parallel transporting the post-selected state $|\Psi'\rangle$ to the pre-selected state's ray $[\Psi]$.

The time integral of (6) can be of value for the study of the geometric quantum phase. As an example of this, consider the product

$$\mathrm{e}^{\mathrm{i}\Omega} = \frac{\langle\Psi'(t_1+\Delta t_1)|\Psi'(t_2+\Delta t_2)\rangle}{|\langle\Psi'(t_1+\Delta t_1)\,|\Psi'(t_2+\Delta t_2)\rangle|} \frac{\langle\Psi'(t_2+\Delta t_2)|\Psi(t_2)\rangle}{|\langle\Psi'(t_2+\Delta t_2)|\Psi(t_2)\rangle|} \frac{\langle\Psi(t_2)|\Psi(t_1)\rangle}{|\langle\Psi(t_2)|\Psi(t_1)\rangle|} \times \frac{\langle\Psi(t_1)|\Psi'(t_1+\Delta t_1)\rangle}{|\langle\Psi(t_1)|\Psi'(t_1+\Delta t_1)\rangle|}.$$

Application of the *horizontal lift theorem* [26] reveals that Ω is the phase acquired by parallel transporting $|\Psi'(t_1+\Delta t_1)\rangle$ along the paths in $\mathcal{H}$ which are the pre-images under Π of the shortest geodesics in $\wp$ joining this state and the rays in the cycle

$$|\Psi'(t_1+\Delta t_1)\rangle \to [\Psi'(t_2+\Delta t_2)] \to [\Psi(t_2)] \to [\Psi(t_1)] \to [\Psi'(t_1+\Delta t_1)]. \tag{7}$$

From (5) it is seen that the second and fourth factors in the product are $\mathrm{e}^{\mathrm{i}\chi(t_2)}$ and $\mathrm{e}^{-\mathrm{i}\chi(t_1)}$, respectively, so that (6) can be used to obtain

$$\mathrm{e}^{\mathrm{i}[\chi(t_2)-\chi(t_1)]} = \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \mathrm{Re}(H' - H)_w\,\mathrm{d}t\right].$$

Substituting this into the product and simplifying the result yields

$$\mathrm{e}^{\mathrm{i}\Omega} = \left[\frac{\langle\Psi'(t_1+\Delta t_1)\,|\Psi'(t_2+\Delta t_2)\rangle}{\langle\Psi'(t_2+\Delta t_2)\,|\Psi'(t_1+\Delta t_1)\rangle} \frac{\langle\Psi(t_2)|\Psi(t_1)\rangle}{\langle\Psi(t_1)|\Psi(t_2)\rangle}\right]^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \mathrm{Re}(H' - H)_w\,\mathrm{d}t\right].$$

It can be concluded from this that

$$\Omega = \left(\frac{1}{\hbar}\right)\int_{t_1}^{t_2} \mathrm{Re}(H' - H)_w \,\mathrm{d}t \tag{8}$$

for pre-selected/post-selected quantum systems for which the factor in brackets is unity.

Before closing this section, it is noted that Aitchison and Wanelik [27] have demonstrated that a complex geometric phase Θ can be defined for quantum systems with two state vectors. Examination of their equation (A6) shows that there is an explicit relationship between Θ and the weak energy associated with a pre-selected/post-selected quantum system. Specifically, it is found that

$$\mathrm{e}^{\mathrm{i}\Theta} = \left[\frac{\langle\Psi'(t_1)|\Psi\,(t_2)\rangle}{\langle\Psi'(t_2)|\Psi(t_1)\rangle}\right]^{1/2} \exp\left[\left(\frac{\mathrm{i}}{2\hbar}\right)\int_{t_1}^{t_2}\left(H' + H\right)_{w_0}\mathrm{d}t\right] \tag{9}$$

where

$$\left(H' + H\right)_{w_0} \equiv \frac{\langle\Psi'(t)|\hat{H}' + \hat{H}|\Psi(t)\rangle}{\langle\Psi'(t)|\Psi(t)\rangle}.$$

Observe that this relationship now involves the *sum of the weak energies* of the post-selected and pre-selected states rather than their difference. Also, as denoted by the subscript w_0, this sum is defined in terms of simultaneous pre-selected and post-selected states, i.e. for $\Delta t = 0$. As before, if the factor in brackets is unity, then

$$\Theta = \left(\frac{1}{2\hbar}\right)\int_{t_1}^{t_2}\left(H' + H\right)_{w_0}\mathrm{d}t.$$

3. Time translators for correlation amplitudes and probabilities

It will now be shown that the weak energy defines a multiplier which relates the correlation amplitude for a quantum system pre-selected at time t_1 and post-selected at $t_1 + \Delta t_1$ to that for a system pre-selected at $t_2 \geqslant t_1$ and post-selected at $t_2 + \Delta t_2$, where Δt_1 and Δt_2 are constants. This multiplier is the weak value at t_1 of a composite time evolution operator for the system which moves the pre-selected state at t_1 to that at t_2 and the post-selected state at $t_2 + \Delta t_2$ to that at $t_1 + \Delta t_1$.

Using (2) it is readily found that the correlation amplitude for a pre-selected/post-selected system obeys the differential equation

$$\frac{\mathrm{d}\langle\Psi'(t')|\Psi(t)\rangle}{\mathrm{d}t} = \left(\frac{\mathrm{i}}{\hbar}\right)(H' - H)_w\langle\Psi'(t')|\Psi(t)\rangle$$

which, upon integration, yields the result

$$\frac{\langle\Psi'(t_2 + \Delta t_2)|\Psi\,(t_2)\rangle}{\langle\Psi'(t_1 + \Delta t_1)|\Psi\,(t_1)\rangle} = \exp\left[\left(\frac{\mathrm{i}}{\hbar}\right)\int_{t_1}^{t_2}(H' - H)_w\,\mathrm{d}t\right]. \tag{10}$$

Let $\hat{H}'$ and $\hat{H}$ be time independent and consider the composite time evolution operator

$$\hat{T}(t_1, t_2) \equiv \hat{U}'^{\dagger}(t_1 + \Delta t_1, t_2 + \Delta t_2)\,\hat{U}(t_1, t_2)$$

where

$$\hat{U}'(t_1 + \Delta t_1, t_2 + \Delta t_2)|\Psi'(t_1 + \Delta t_1)\rangle = \exp\left[-\left(\frac{\mathrm{i}}{\hbar}\right)\hat{H}'(t_2 + \Delta t_2 - t_1 - \Delta t_1)\right]|\Psi'(t_1 + \Delta t_1)\rangle$$
$$= |\Psi'(t_2 + \Delta t_2)\rangle$$

and

$$\hat{U}(t_1, t_2)|\Psi(t_1)\rangle = \exp\left[-\left(\frac{\mathrm{i}}{\hbar}\right)\hat{H}(t_2 - t_1)\right]|\Psi(t_1)\rangle = |\Psi(t_2)\rangle.$$

Now define the weak value at t_1 for this composite operator in the usual manner:

$$(T(t_1, t_2))_w \equiv \frac{\langle\Psi'(t_1 + \Delta t_1)|\hat{T}(t_1, t_2)|\Psi(t_1)\rangle}{\langle\Psi'(t_1 + \Delta t_1)|\Psi(t_1)\rangle}$$

where

$$\langle\Psi'(t_1 + \Delta t_1)|\Psi(t_1)\rangle \neq 0.$$

This obviously yields

$$(T(t_1, t_2))_w = \frac{\langle\Psi'(t_2 + \Delta t_2)|\Psi(t_2)\rangle}{\langle\Psi'(t_1 + \Delta t_1)|\Psi(t_1)\rangle}$$

and may be rewritten using (10) as

$$(T(t_1, t_2))_w = \exp\left[\left(\frac{\mathrm{i}}{\hbar}\right)\int_{t_1}^{t_2}(H' - H)_w\,\mathrm{d}t\right] \tag{11}$$

or equivalently

$$(T(t_1, t_2))_w = \exp\left[\mathrm{i}\int_{t_1}^{t_2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}t}\right)\mathrm{d}t - \left(\frac{1}{\hbar}\right)\int_{t_1}^{t_2} L(s, \dot{s})\,\mathrm{d}t\right].$$

Thus,

$$\langle\Psi'(t_2 + \Delta t_2)|\Psi(t_2)\rangle = (T(t_1, t_2))_w\,\langle\Psi'(t_1 + \Delta t_1)|\Psi(t_1)\rangle$$

and (11) serves as a multiplier which relates correlation amplitudes in time. Here the real (i.e. $(\mathrm{d}\chi/\mathrm{d}t)$) and imaginary (i.e. $L(s, \dot{s})$) parts of the integrand of (11) capture and transfer from one amplitude to another the essence of the associated temporal changes in Hilbert space dynamics in $\wp$ and the phase acquired from parallel transport in $\mathcal{H}$, respectively. Observe that when (8) applies, this phase is that which results from parallel transport of the post-selected state along cycle (7) as described in the previous section. It is also interesting to note that the indefinite form for (11) is the integrating factor for the general equation of motion for the weak value of a quantum mechanical observable (e.g. equation (3.7) in Parks *et al* [18]).

It is easily seen using (11) that the complex phase Θ in (9) vanishes when $\hat{H} = 0$ or $\hat{H}' = 0$. Specifically, if $\Delta t_1 = 0 = \Delta t_2$, then

$$\left[\frac{\langle\Psi'(t_1)|\Psi(t_2)\rangle}{\langle\Psi'(t_2)|\Psi(t_1)\rangle}\right]^{1/2} = \begin{cases} (T(t_1, t_2))_{w_0}^{-1/2} = \exp\left[-\dfrac{\mathrm{i}}{2\hbar}\displaystyle\int_{t_1}^{t_2}(H')_{w_0}\,\mathrm{d}t\right] & \hat{H} = 0 \\ (T(t_1, t_2))_{w_0}^{1/2} = \exp\left[-\dfrac{\mathrm{i}}{2\hbar}\displaystyle\int_{t_1}^{t_2}(H)_{w_0}\,\mathrm{d}t\right] & \hat{H}' = 0. \end{cases}$$

In either case, the right-hand side of (9) is unity and $\Theta = 0$.

The square modulus of (11) defines another multiplier $\Lambda_w(t_1, t_2)$ that translates correlation probabilities in time. In particular,

$$|\langle\Psi'(t_2 + \Delta t_2)|\Psi(t_2)\rangle|^2 = \Lambda_w(t_1, t_2)|\langle\Psi'(t_1 + \Delta t_1)|\Psi(t_1)\rangle|^2$$

where

$$\Lambda_w(t_1, t_2) = |(T(t_1, t_2))_w|^2 = \exp\left[-\left(\frac{2}{\hbar}\right)\int_{t_1}^{t_2}\mathrm{Im}(H' - H)_w\,\mathrm{d}t\right] \tag{12}$$

or equivalently

$$\Lambda_w(t_1, t_2) = \exp\left[-\left(\frac{2}{\hbar}\right)\int_{t_1}^{t_2} L(s, \dot{s})\,\mathrm{d}t\right].$$

Unlike the case for $(T(t_1, t_2))_w$, $\Lambda_w(t_1, t_2)$ transfers only the $\wp$-related dynamical information; as expected, the parallel transport phase information is lost.

4. Correlation control and weak stationarity

It is implicit from the previous discussion that the correlations for systems that are both pre-selected and post-selected can be controlled by manipulating the associated weak energy. The quantity $\Lambda_w(t_1, t_2)$ can be used to provide a relative measure of the correlations for such systems at different times. In particular, if

$$\mathrm{Im}(H' - H)_w \equiv \Upsilon$$

is continuous on the closed interval $[t_1, t_2]$, then, according to the *first mean-value theorem for integrals*, there is a time ξ in the open interval (t_1, t_2) for which

$$\int_{t_1}^{t_2} \mathrm{Im}(H' - H)_w\,\mathrm{d}t = (t_2 - t_1)\Upsilon_\xi$$

where Υ_ξ is Υ evaluated at $t = \xi$. Application of this allows (12) to be rewritten as

$$\Lambda_w(t_1, t_2) = \exp\left[-\frac{2\Upsilon_\xi}{\hbar}(t_2 - t_1)\right].$$

Here it can be seen that for $\Upsilon_\xi > 0$ ($\Upsilon_\xi < 0$) the correlation for a quantum system pre-selected at t_2 and post-selected at $t_2 + \Delta t_2$ will have *decayed (grown)* appreciably from one pre-selected at t_1 and post-selected at $t_1 + \Delta t_1$ only when

$$t_2 - t_1 \geqslant \frac{\hbar}{2|\Upsilon_\xi|} \tag{13}$$

where the ratio $\hbar/(2|\Upsilon_\xi|)$ is the *mean lifetime* for the correlation in the interval $[t_1, t_2]$.

A system is *weakly stationary* when the mean lifetime for the associated correlation is infinite. Then the correlation amplitudes for systems pre-selected at the end points of the interval $[t_1, t_2]$ will differ at most by a phase factor so that $\Lambda_w(t_1, t_2) = 1$. There is then no decay or growth associated with the correlations at these times. As can be seen from (13), this obviously occurs when ξ is a t intercept for the graph of Υ so that Υ_ξ vanishes (e.g. when ξ is the midpoint for the interval $[t_1, t_2]$ and the graph of Υ is symmetric with respect to ξ in $[t_1, t_2]$). *Trivial stationarity* exists when $(H)_w = 0 = (H')_w$ during $[t_1, t_2]$ so that $\Upsilon_\xi = 0$ for all ξ in the interval. Such a system is weakly stationary and the following implication holds:

$$\text{trivial stationarity} \Rightarrow \text{weak stationarity}.$$

When both $|\Psi\rangle$ and $|\Psi'\rangle$ are stationary, then the system is *bi-stationary*. In this case, $\Upsilon = 0$ for all t. This is clearly a strong form of weak stationarity so that the implication

$$\text{bi-stationarity} \Rightarrow \text{weak stationarity}$$

is valid.

Before concluding this section, it is noted that the existence of trivial stationarity during an interval $[t_1, t_2]$ provides an interesting ratio equivalence between the values at t_1 and t_2

for non-diagonal matrix elements for an observable $\hat{A}$ and the associated weak values $A_w(t)$, where

$$A_w(t) = \frac{\langle \Psi'(t+\Delta t)|\hat{A}|\Psi(t)\rangle}{\langle \Psi'(t+\Delta t)|\Psi(t)\rangle}.$$

This follows directly from the fact that if trivial stationarity prevails during a time interval $[t_1, t_2]$, then

$$(T(t_1,t_2))_w = 1 \quad \Rightarrow \quad \langle \Psi'(t_1+\Delta t_1)|\Psi(t_1)\rangle = \langle \Psi'(t_2+\Delta t_2)|\Psi(t_2)\rangle.$$

Straightforward application of this equality to the last equation yields

$$\frac{A_w(t_2)}{A_w(t_1)} = \frac{\langle \Psi'(t_2+\Delta t_2)|\hat{A}|\Psi(t_2)\rangle}{\langle \Psi'(t_1+\Delta t_1)|\hat{A}|\Psi(t_1)\rangle}.$$

Thus, the ratio on the right (left) side of the equality serves as a multiplier which moves the weak value (non-diagonal matrix element) from its t_1 value to its value at t_2.

5. Weak energy dynamics: a special case

As was seen in section 3, the weak energy is responsible for the temporal changes in the correlation amplitude. For this reason, the dynamical characteristics of $(H'-H)_w$ are of general interest. These characteristics can be obtained from an examination of the properties of its equation of motion.

The equation of motion for the weak energy is readily obtained by using (2) in the time derivative of (1) and is given by

$$\frac{\mathrm{d}(H'-H)_w}{\mathrm{d}t} = \left(\frac{\mathrm{i}}{\hbar}\right)\left\{\left(H'^2 - 2H'H + H^2\right)_w - (H'-H)_w^2\right\} + \left(\frac{\mathrm{d}(\hat{H}'-\hat{H})}{\mathrm{d}t}\right)_w.$$

When this derivative vanishes, then $(H'-H)_w = \gamma$ is a *weak constant of the motion* and γ is a *good weak quantum number*.

Consider now the special case where the commutator $[\hat{H}, \hat{H}'] = 0$ and $\mathrm{d}\hat{H}/\mathrm{d}t = \mathrm{d}\hat{H}'/\mathrm{d}t = 0$, i.e. when $\hat{H}$ and $\hat{H}'$ are *mutual constants of the motion*. Then the last equation may be written as

$$\mathrm{i}\hbar\frac{\mathrm{d}(H'-H)_w}{\mathrm{d}t} = -\Delta_w^2(H'-H) \tag{14}$$

where

$$\Delta_w^2(H'-H) = \left\{(H'-H)^2\right\}_w - (H'-H)_w^2$$

is the weak variance for the weak energy. The *weak energy uncertainty* [9] is related to this weak variance via

$$\Delta(H'-H)_w = \left|\Delta_w^2(H'-H)\right|^{1/2}.$$

Using (14) in this equation yields

$$\Delta(H'-H)_w = \hbar^{1/2}\left|\frac{\mathrm{d}(H'-H)_w}{\mathrm{d}t}\right|^{1/2}. \tag{15}$$

Let τ represent the characteristic time required for the weak energy to be changed by an amount equal to the width $\Delta(H' - H)_w$ of its statistical distribution. Then, in general,

$$\tau = \frac{\Delta(H' - H)_w}{|(\mathrm{d}(H' - H)_w)/\mathrm{d}t|}.$$

This suggests that appreciable differences between correlation amplitudes should be expected only for times with differences significantly greater than τ. Also, it would be anticipated that τ is generally infinite when $(H' - H)_w$ is a good weak quantum number. This would imply that the amplitude is weakly stationary.

However, substitution of the square of (15) into the denominator of the last equation shows that when $\hat{H}$ and $\hat{H}'$ are mutual constants of the motion, then τ must adhere to the following *time-weak energy uncertainty relation*:

$$\tau\, \Delta(H' - H)_w = \hbar.$$

When these conditions prevail, this equation may be used to rewrite (13) as

$$t_2 - t_1 \geqslant \frac{\tau\, \Delta(H' - H)_w}{2|\Upsilon_\xi|}. \tag{16}$$

Hence, under these conditions and regardless of the value of τ, weak stationarity can exist only when Υ_ξ vanishes. Thus, when $(H' - H)_w$ is a good weak quantum number (15) vanishes and τ is infinite so that the numerator for (16) remains constant as required and weak stationarity does not generally exist. This is easily seen by letting $(H' - H)_w = \gamma_0 + \mathrm{i}\gamma_1$, where γ_0 and γ_1 are fixed real numbers. Then $\Upsilon_\xi = \gamma_1$ and the right-hand side of (16) is finite unless $\gamma_1 = 0$. Observe that this is precisely the situation that occurs for trivially stationary systems and bi-stationary systems. There the Hamiltonians are obviously mutual constants of the motion and the associated weak energy is a good weak quantum number with $\gamma_1 \equiv \mathrm{Im}(H' - H)_w = 0 = \mathrm{Im}(H')_w$.

6. A two state example

For the purpose of illustrating the results developed above, they will be applied to a simple two state system with $\Delta t = \kappa$, κ a constant. Let

$$|\Psi'(t + \Delta t)\rangle = |\Psi'(t + \kappa)\rangle = \cos\omega\,(t + \kappa)\,|\varphi\rangle + \sin\omega\,(t + \kappa)\,|\varphi'\rangle$$

where $\langle\varphi|\varphi'\rangle = 0$ and ω is a constant. Assume that

$$|\Psi(t)\rangle = \cos\beta_0|\varphi\rangle + \mathrm{i}\sin\beta_0|\varphi'\rangle$$

with β_0 constant, i.e. $|\Psi\rangle$ is fixed for all time, so that $\hat{H} = 0$. Using (2) it can be determined that $\hat{H}' = \hbar\omega\hat{\sigma}_y$, where $\hat{\sigma}_y$ is the usual Pauli spin operator. Hence, $\hat{H}$ and $\hat{H}'$ are mutual constants of the motion. Then from (1),

$$\mathrm{Re}(H' - H)_w = \mathrm{Re}(H')_w = \frac{\hbar\omega\sin 2\beta_0}{2\left\{\cos^2\beta_0 - \cos 2\beta_0 \sin^2\omega\,(t + \kappa)\right\}}$$

and

$$\mathrm{Im}(H' - H)_w = \mathrm{Im}(H')_w = \frac{\hbar\omega\,\{\cos 2\beta_0 \sin\omega\,(t + \kappa)\cos\omega\,(t + \kappa)\}}{\left\{\cos^2\beta_0\cos^2\omega\,(t + \kappa) + \sin^2\beta_0 \sin^2\omega\,(t + \kappa)\right\}}.$$

If $t_1 = 0$ and $t_2 = \zeta$, then (11) and (12) yield

$$(T(0,\zeta))_w = \left[\frac{\cos^2\beta_0 - \cos 2\beta_0 \sin^2\omega(\zeta+\kappa)}{\cos^2\beta_0 - \cos 2\beta_0 \sin^2\omega\kappa}\right]^{1/2} \times \exp\left[\mathrm{i}\left\{\tan^{-1}[\tan\beta_0 \tan\omega(\zeta+\kappa)] - \tan^{-1}[\tan\beta_0\tan\omega\kappa]\right\}\right]$$

and

$$\Lambda_w(0,\zeta) = \frac{\cos^2\beta_0 - \cos 2\beta_0 \sin^2\omega(\zeta+\kappa)}{\cos^2\beta_0 - \cos 2\beta_0 \sin^2\omega\kappa}.$$

Thus, as required,

$$\begin{aligned}\langle\Psi'(\zeta+\kappa)|\Psi(\zeta)\rangle &= (T(0,\zeta))_w\langle\Psi'(\kappa)|\Psi(0)\rangle \\ &= \left[\cos^2\beta_0 - \cos 2\beta_0\sin^2\omega(\zeta+\kappa)\right]^{1/2}\exp\left[\mathrm{i}\tan^{-1}[\tan\beta_0\tan\omega(\zeta+\kappa)]\right]\end{aligned}$$

and

$$\begin{aligned}\left|\langle\Psi'(\zeta+\kappa)|\Psi(\zeta)\rangle\right|^2 &= \Lambda_w(0,\zeta)\left|\langle\Psi'(\kappa)|\Psi(0)\rangle\right|^2 \\ &= \cos^2\beta_0 - \cos 2\beta_0\sin^2\omega(\zeta+\kappa).\end{aligned}$$

The following observations are of pedagogical value:

(a) Here (8) applies and

$$\Omega = \tan^{-1}[\tan\beta_0\tan\omega(\zeta+\kappa)] - \tan^{-1}[\tan\beta_0\tan\omega\kappa]$$

is the phase acquired by parallel transporting $|\Psi'(\kappa)\rangle$ in the system's Hilbert space as prescribed by cycle (7). Note that in this case $|\Psi(t_1)\rangle = |\Psi(t_2)\rangle$ so that the image of this cycle in the associated projective Hilbert space follows geodesics that are the sides of a spherical triangle on the Poincaré sphere.

(b) Since $\hat{H} = 0$, the complex phase Θ in (9) vanishes when $\kappa = 0$.

(c) Correlation decay can be induced into this two state system by manipulating the weak energy parameters ω and κ associated with the post-selected state. In particular,

$$\begin{aligned}\Lambda_w(0,\zeta) < 1 \quad &\Rightarrow \quad \cos^2\beta_0 - \cos 2\beta_0\sin^2\omega(\zeta+\kappa) < \cos^2\beta_0 - \cos 2\beta_0\sin^2\omega\kappa \\ &\Rightarrow \quad \sin^2\omega(\zeta+\kappa) - \sin^2\omega\kappa > 0 \Rightarrow \sin\omega(\zeta+2\kappa)\sin\omega\zeta > 0 \\ &\Rightarrow \quad \text{correlation decay exists when } \frac{2\pi k}{\omega} < \zeta < \frac{(2k+1)\pi}{\omega} - 2\kappa\end{aligned}$$

or

$$\frac{(2k+1)\pi}{\omega} < \zeta < \frac{2(k+1)\pi}{\omega} - 2\kappa \qquad \text{with} \quad 2\kappa < \frac{\pi}{\omega}$$

where k is a non-negative integer.

(d) Correlation growth can also be induced by manipulating ω and κ so that

$$\frac{2(k+1)\pi}{\omega} - 2\kappa < \zeta < \frac{2(k+1)\pi}{\omega}$$

or

$$\frac{(2k+1)\pi}{\omega} - 2\kappa < \zeta < \frac{(2k+1)\pi}{\omega} \qquad \text{with} \quad \kappa \neq 0$$

where k is a non-negative integer.

(e) The system can be made weakly stationary through proper selection of the weak energy parameters ω and κ associated with the post-selected state. Specifically,

$$\Lambda_w(0,\zeta)=1 \quad \Rightarrow \quad \text{the system is weakly stationary when } \zeta=\frac{k\pi}{\omega}$$

where k is a positive integer.

(f) For any t and k a non-negative integer,

$$\beta_0=\tfrac{1}{4}(2k+1)\pi \quad \Rightarrow \quad \mathrm{Re}(H'-H)_w=\pm\hbar\omega \qquad \text{and} \qquad \mathrm{Im}(H'-H)_w=0$$
$$\Rightarrow \quad (H'-H)_w \text{ is a good weak quantum number}$$

and

$$\mathrm{Im}(H'-H)_w=0 \quad \Rightarrow \quad \Upsilon=\Upsilon_\xi=0 \quad \Rightarrow \quad \text{the system is weakly stationary.}$$

(g) For any t the system conforms to the time-weak energy uncertainty relation. This can be easily confirmed for selected values of β_0 from the following independent calculations:

$$\beta_0=0,\pi \quad \Rightarrow \quad \mathrm{Re}(H'-H)_w=0 \qquad \text{and} \qquad \mathrm{Im}(H'-H)_w=\hbar\omega\tan\omega(t+\kappa)$$
$$\Rightarrow \quad \Delta_w^2(H'-H)=\hbar^2\omega^2\sec^2\omega(t+\kappa)$$
$$\Rightarrow \quad \Delta(H'-H)_w=\hbar\omega\sec\omega(t+\kappa)$$
$$\Rightarrow \quad \tau=[\omega\sec\omega(t+\kappa)]^{-1}$$

since

$$\left|\frac{\mathrm{d}(H'-H)_w}{\mathrm{d}t}\right|=\hbar\omega^2\sec^2\omega(t+\kappa) \quad \Rightarrow \quad \tau\,\Delta(H'-H)_w=\hbar.$$

Similarly,

$$\beta_0=\tfrac{1}{2}\pi,\tfrac{3}{2}\pi \quad \Rightarrow \quad \mathrm{Re}(H'-H)_w=0 \qquad \text{and} \qquad \mathrm{Im}(H'-H)_w=-\hbar\omega\cot\omega(t+\kappa)$$
$$\Rightarrow \quad \Delta_w^2(H'-H)=\hbar^2\omega^2\,\mathrm{cosec}^2\,\omega(t+\kappa)$$
$$\Rightarrow \quad \Delta(H'-H)_w=\hbar\omega\,\mathrm{cosec}\,\omega(t+\kappa)$$
$$\Rightarrow \quad \tau=[\omega\,\mathrm{cosec}\,\omega(t+\kappa)]^{-1}$$

since

$$\left|\frac{\mathrm{d}(H'-H)_w}{\mathrm{d}t}\right|=\hbar\omega^2\,\mathrm{cosec}^2\,\omega(t+\kappa) \quad \Rightarrow \quad \tau\,\Delta(H'-H)_w=\hbar.$$

7. The dynamic quantum eraser: time-dependent *welcher Weg* information and weak stationarity

In order to provide additional clarity concerning the previous results, they will be applied to a specific pre-selection/post-selection system-the 'dynamic quantum eraser'. This apparatus is identical to the 'static' quantum eraser discussed by Kwiat *et al* [28], except that the two linear photon polarization filters are rotating with angular speeds ω_1 and ω_2, respectively. Here, two beams of identical linearly polarized conjugate photon pairs are produced from pump photons via a type-I down-conversion process. These beams traverse distinct paths of equal length and (after a half-wave plate orthogonal rotation of the polarization states of the photons in one of the beams) converge upon the input ports of a 50/50 beamsplitter. There are two paths through the beamsplitter which can result in coincidence counts at the photon detectors (i.e.

both photons are transmitted or both are reflected). Their indistinguishability is responsible for the loss of which path (*welcher Weg*) photons follow through the apparatus.

In the absence of the filters the orthogonal polarization states 'tag' photons and provide the welcher Weg information necessary to obviate the fourth-order interference in coincidence count (observed as a null in the count) that would occur had the orthogonal rotation ('tagging') not been performed on the photons in one of the beams. When non-rotating polarization filters are inserted in the paths between the output ports of the beamsplitter and the photon detectors, they can be used to 'erase' the welcher Weg information resident in the polarization states and restore the interference null in coincidence count. Then the coincidence count profile (i.e. the coincidence count probability) for the static quantum eraser is given by $\frac{1}{4}\sin^2(\theta_2-\theta_1)$, where θ_1 and θ_2 are the fixed angular settings for the filters.

The quantum eraser is clearly a pre-selection/post-selection apparatus for conjugate photon pairs. The down-conversion process, when coupled with the half-wave plate and beamsplitter, pre-selects entangled photon pair polarization and number states given by

$$|\Psi(t)\rangle = \tfrac{1}{2}\left\{|1_1^H 1_2^V\rangle - |1_1^V 1_2^H\rangle\right\}$$

where only those terms related to coincidence count have been retained. Here, the superscripts denote the polarization state (i.e. $H=$ 'horizontal' and $V=$ 'vertical'), subscripts denote the beamsplitter output ports and t refers to the time at which the photon pair 'exits' from the beamsplitter. When the filters are rotating in the dynamic quantum eraser, they post-select the tensor product state

$$\begin{aligned}|\Psi'(t+\delta)\rangle = &\left\{\cos\left[\omega_1(t+\delta)+\theta_1\right]|1_1^H\rangle + \sin\left[\omega_1(t+\delta)+\theta_1\right]|1_1^V\rangle\right\}\\ &\times\left\{\cos\left[\omega_2(t+\delta)+\theta_2\right]|1_2^H\rangle + \sin\left[\omega_2(t+\delta)+\theta_2\right]|1_2^V\rangle\right\}\end{aligned}$$

where δ is the constant difference between pre-selection and post-selection times. Clearly, the dynamic quantum eraser produces time-dependent welcher Weg information.

It is apparent that for this apparatus $\hat{H}=0$. Also, using (2) it can be easily determined that

$$\hat{H}' = \hbar\omega_1\begin{pmatrix}\tilde{0} & \tilde{\sigma}_y\\ \tilde{\sigma}_y & \tilde{0}\end{pmatrix} - \mathrm{i}\hbar\omega_2\begin{pmatrix}\tilde{0} & \tilde{\sigma}_z\\ -\tilde{\sigma}_z & \tilde{0}\end{pmatrix}$$

where $\tilde{0}$ is a 2×2 zero matrix and $\tilde{\sigma}_y$ and $\tilde{\sigma}_z$ are the usual 2×2 Pauli spin matrices (the assumed matrix order for the post-selection basis states is $|1_1^H 1_2^H\rangle, |1_1^V 1_2^V\rangle, |1_1^H 1_2^V\rangle, |1_1^V 1_2^H\rangle$). From these, the weak energy for the system is readily found to be

$$(H'-H)_w = (H')_w = \mathrm{i}\hbar(\omega_2-\omega_1)\cot[(\omega_2-\omega_1)t + (\omega_2-\omega_1)\delta + (\theta_2-\theta_1)].$$

Observe that the system is weakly stationary for time intervals $[t_-, t_+]$, where

$$t_\pm + \delta = \frac{(2k_\pm+1)\,\pi - 2\,(\theta_2-\theta_1) \pm 2\varphi}{2(\omega_2-\omega_1)} \qquad 0\leqslant\varphi<\tfrac{1}{2}\pi \quad k_\pm = 0,1,2,\ldots \quad k_-\leqslant k_+.$$

Then the coincidence count probabilities C given in general by

$$C = |\langle\Psi'|\Psi\rangle|^2 = \tfrac{1}{4}\sin^2\left[(\omega_2-\omega_1)t + (\omega_2-\omega_1)\delta + (\theta_2-\theta_1)\right]$$

for photon pairs post-selected at $t_\pm+\delta$ are equal with value $\frac{1}{4}\cos^2\varphi$. Equivalent welcher Weg information is therefore produced by the system at these times.

Since $\hat{H}'$ and $\hat{H}$ are mutual constants of the motion and δ is fixed, the system conforms to the time-weak energy uncertainty relation with

$$\Delta(H'-H)_w = \hbar\,(\omega_2-\omega_1)\,\mathrm{cosec}[(\omega_2-\omega_1)t + (\omega_2-\omega_1)\delta + (\theta_2-\theta_1)].$$

If $\omega_1 \neq \omega_2$, then $C = 0$ (i.e. all welcher Weg information is lost) when

$$t + \delta = \frac{k\pi - (\theta_2 - \theta_1)}{\omega_2 - \omega_1} \qquad k = 0, 1, 2 \ldots.$$

Note that in this case, the complete erasure of welcher Weg information occurs exactly at those times for which the weak energy and its uncertainty are infinite (and the pre-selected and post-selected states are orthogonal).

Also of interest is the case where $\omega_1 = \omega_2 = \omega$ so that there is no time-dependent welcher Weg information produced by the rotating filters. Then for $\theta_1 \neq \theta_2$, $(H' - H)_w = (H')_w = 0$ is a good weak quantum number and the system is trivially stationary (with $\Delta(H' - H)_w = 0$ and $\tau \to \infty$) so that C is constant in time with value

$$C = \tfrac{1}{4}\sin^2(\theta_2 - \theta_1).$$

Thus, the coincidence counts (and associated welcher Weg information) produced by a trivially stationary dynamic quantum eraser and a static quantum eraser with the same initial filter settings are identical.

8. Concluding remarks

This paper has provided a description of weak energy in terms of its associated Hilbert space geometry and dynamics. Its general significance for pre-selected/post-selected systems has also been discussed in terms of a weak stationarity condition for the associated correlation probabilities and a time-weak energy uncertainty relation. Applications of the theory to a general two state quantum system and the 'dynamic quantum eraser' suggest that techniques developed from this theory may be of value for the temporal control of quantum correlations, as well as for the experimental measurement of related kinematic parameters. Finally, since the definite time integral of the real part of the weak energy provides a measure of the difference in parallel transport phase acquired in Hilbert space at distinct times, this aspect of the theory should also prove useful for the study of geometric quantum phase.

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